MATH 2411 T1B Tutorial 2

HOU Zhen

zhouah@connect.ust.hk

February 24, 2026

Basic Probability

Basic Concepts

- **Random Experiment/Trial**: A procedure that generates an uncertain outcome
- **Sample Space**: The set of all possible outcomes of an experiment, denoted by Ω
- **Event**: A set of outcomes

Operation rules of events

- **Union**: $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- **Intersection**: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- **Complement**: $A^c = \{x : x \notin A\}$.
- **Difference**: $A - B = \{x : x \in A \text{ and } x \notin B\}$.
- **Symmetric Difference**: $A \triangle B = (A - B) \cup (B - A)$.

Example. $\Omega = \{1, 2, 3, 4\}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$.

- $A \cup B = \{1, 2, 3, 4\}$.
- $A \cap B = \{2, 3\}$.
- $A^c = \{4\}$.
- $A - B = \{1\}$.
- $A \triangle B = \{1, 4\}$.

Operation rules of events

- **Commutative**: $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- **Associative**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
- **Distributive**: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- (For the above three rules, note the similarity between $(\cup, \cap)$ and $(+, \times)$.)
- **De Morgan's Laws**: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Useful Concepts

- **Empty event**: $\emptyset$
- **Disjoint events**: $A \cap B = \emptyset$
- **Mutually exclusive** events $A_1, A_2, \ldots, A_k$ if any pair of them are disjoint.
- **Exhaustive** events $A_1, A_2, \ldots, A_n$ if the union $\bigcup_{i=1}^{n} A_i = \Omega$.

Definition of Probability

A function P : Events in $\Omega \to \mathbb{R}$ is a **probability function** if it satisfies the following axioms:

1. $0 \leq P(E) \leq 1$ for any event E. (Non-negativity)
2. $P(\Omega) = 1$. (Normalization)
3. If $E_1, E_2, \ldots$ are mutually exclusive events, then $P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots$. (Additivity)

A sample space Ω together with a collection of events $\mathcal{A}$ and a probability function P is called a **probability space**: $(\Omega, \mathcal{A}, P)$.

Properties of Probability

1. $P(A^c) = 1 - P(A)$.
2. $P(\emptyset) = 0$.
3. If $A \subset B$, then $P(A) \leq P(B)$.
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
5. As $P(A \cap B) \geq 0$, we have $P(A \cup B) \leq P(A) + P(B)$.
6. Moreover, with De Morgan's Law, we have $P(A^c \cap B^c) \geq 1 - P(A) - P(A)$.
7. $P(A \cup B \cup C) =$ $P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Exercise

Suppose that $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$.

1. Find $P(A^c)$.
2. Find $P(A \cup B)$.
3. Find $P(A^c \cap B^c)$.

Sample Space with Equally Likely Outcomes

$S = \{\omega_1, \omega_2, \ldots, \omega_N\}$ is called a **sample space with equally likely outcomes** if $P(\{\omega_i\}) = P(\{\omega_j\})$ for all i, j.
For any event E, we have

$$P(E) = \frac{|E|}{|S|},$$

where $|E|$ denotes the number of elements in E.

Counting Techniques-Addition Principle

$$n \text{ different WAYS } \begin{cases} \text{There are } m_1 \text{ outcomes from the 1st way} \\ \text{There are } m_2 \text{ outcomes from the 2nd way} \\ \cdots\cdots\cdots \\ \text{There are } m_n \text{ outcomes from the } n\text{-th way} \end{cases}$$

The total number of outcomes is $m_1 + m_2 + \cdots + m_n$.

Counting Techniques—Multiplication Principle

$$n \text{ different STEPS} \begin{cases} \text{There are } m_1 \text{ outcomes in the 1st step} \\ \text{There are } m_2 \text{ outcomes in the 2nd step} \\ \cdots\cdots\cdots \\ \text{There are } m_n \text{ outcomes in the } n\text{-th step} \end{cases}$$

The total number of outcomes is $m_1 \times m_2 \times \cdots \times m_n$.

Example: If there are n_1 ways to do the first task, and for each of these ways there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the two tasks.

Example: To choose a shirt and a pair of pants from 5 shirts and 4 pairs of pants, there are $5 \times 4 = 20$ ways.

Exercise

Two people are playing a game with a fair coin. They keep flipping the coin repeatedly until someone wins. If the outcome sequence "tails–heads–heads" appears, then the first person wins; if the outcome sequence "heads–heads–tails" appears, then the second person wins. Is this game fair?

Counting Techniques-Permutation

A permutation is an ordered arrangement of all or part of a set of objects.

The number of permutations of n objects is $n!$.

The number of permutations of n objects taken r at a time is $\frac{n!}{(n-r)!}$.
Example: To put 5 different books on a shelf, there are $5! = 120$ ways.

The number of ways to choose k objects from n distinct objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
Example: To choose 3 books from 5 different books, there are $\binom{5}{3} = 10$ ways.

Exercise

Suppose that there are 3 red balls and 5 blue balls in a box. If we randomly choose 2 balls from the box, what is the probability that the two balls are of different colors?

Conditional Probability

If $P(B) > 0$, the **conditional probability** of event A given event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note: If $P(B) > 0$, then $P(A|B)$ is a probability function on the sample space B.

Exercise

Suppose that a box contains 3 red balls and 5 blue balls. If we randomly choose 2 balls by order from the box, what is the probability that the two balls are of different colors given that the first ball is red?

Independence

Two events A and B are called **independent** if $P(A \cap B) = P(A)P(B)$.

If $P(B) > 0$, by the definition of conditional probability, we can find that $P(A \cap B) = P(B)P(A|B)$. Thus, A and B are independent if and only if $P(A|B) = P(A)$, which can be interpreted as the occurrence of B does not affect the probability of A.

Law of Total Probability

Suppose that $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, i.e., $\bigcup_{i=1}^{n} B_i = \Omega$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Then for any event A, we have

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

Exercise

Suppose that a box contains 3 red balls and 5 blue balls. We randomly choose 2 balls by order from the box. Let B be the event that the first ball is red, and A be the event that the second ball is red. Are A and B independent?