MATH 2411 T1B Tutorial 3

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Basic Probability

Bayes' Theorem

If $P(A) > 0$ and $P(B) > 0$, then

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}.$$

Note:

1. Bayes' theorem is a direct consequence of the definition of conditional probability.
2. $P(B)$ is called prior probability and $P(B|A)$ is called posterior probability.

A Further Expression with Law of Total Probability

With the law of total probability, we can also write Bayes' theorem as

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}.$$

Further Generalization

If $B_1, B_2, \ldots, B_k$ is a partition and $P(B_i) > 0$ for all i, then

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}.$$

Exercise

True positive rate means the probability of testing positive given that the person has the disease, and false positive rate means the probability of testing positive given that the person does not have the disease.
For a disease, a test has a 95% true positive rate and a 10% false positive rate. The disease occurs in 1% of the population. If a person tests positive, what is the probability that he has the disease?

Exercise

When coded messages are sent, there may be errors in the transmission. In particular, Morse code used "dots" and "dashes", which are known to occur in the proportion of dot:dash = 3:4. Suppose there is interference on the transmission line with probability $1/8$ that a dot is incorrectly received as a dash, and probability $1/8$ that a dash is incorrectly received as a dot. Now, a single signal is randomly selected and then sent to us. If we receive a dot, then what is the probability that the actual signal sent to us is a dot?

Random Variable

Definition of a Random Variable

A random variable (r.v.) is a **function** X on the sample space Ω that associates a real number $X(\omega)$ with each element(outcome) ω in the sample space Ω.

Two types of random variables: discrete and continuous.

Discrete Random Variable

A random variable X is called **discrete** if it takes finite or countably infinite number of possible values.

Probability Mass Function

The probability mass function (pmf) of a discrete random variable X is the function $p_X(x) = \mathbb{P}(X = x)$, for all x in R_X (the range of X).

Properties:

1. $0 \le p_X(x) \le 1$ for all x in R_X.
2. $\sum_{x \in R_X} p_X(x) = 1$.

Note: A function satisfying the above two properties also defines a valid pmf of a random variable.

Cumulative Distribution Function

The cumulative distribution function (cdf) of a **random variable** X is the function $F_X(x) = \mathbb{P}(X \leq x)$, for $x \in \mathbb{R}$.

Properties of discrete cdf:

1. $0 \leq F_X(x) \leq 1$ for all x in $\mathbb{R}$.
2. $F_X(x)$ is non-decreasing.
3. $F(-\infty) := \lim_{x \to -\infty} F_X(x) = 0$ and $F(+\infty) := \lim_{x \to +\infty} F_X(x) = 1$.
4. $F_X(x)$ is right-continuous, i.e., $\lim_{x \to a^+} F_X(x) = F_X(a)$ for all $a \in \mathbb{R}$.
5. $\mathbb{P}(a < X \leq b) = F_X(b) - F_X(a)$ for all $a, b \in \mathbb{R}$.

Relation between pmf and cdf for discrete random variables

$$F_X(a) = P(X \le a) = \sum_{y \le a} p_X(y)$$

If the range of X is expressed by $\{x_1, x_2, \ldots\}$, where $x_1 < x_2 < \ldots$, then

$$\begin{aligned} p_X(x_1) &= F_X(x_1), \\ p_X(x_j) &= F_X(x_j) - F_X(x_{j-1}), \quad j = 2, 3, \ldots \end{aligned}$$

Exercise

Tossing a fair coin three times, let X be the number of heads in these tosses. Calculate the pmf and cdf of X.

How about n tosses?

Population Mean

The (population) mean or expectation of a discrete random variable X is defined as

$$\mathbb{E}[X] = \sum_{x \in R_X} x p_X(x)$$

if $\sum_{x \in R_X} |x| p(x) < +\infty$ which guarantees that $\mathbb{E}[X]$ exists.

Exercise

Find C such that $\mathbb{E}[(X - C)^2] = \sum_{x \in R_X} (x - C)^2 p_X(x)$ is minimized.