MATH 2411 T1B Tutorial 6

HOU Zhen

zhouah@connect.ust.hk

March 24, 2026

Method of Moments Estimation

- Estimate unknown parameters $\theta_1, \ldots, \theta_m$ by matching **theoretical moments** to **sample moments**.
- Population moments depend on the parameters:

$$E(X^k) = \alpha_k(\theta_1, \ldots, \theta_m), \quad k = 1, \ldots, m.$$

- Solve this system for the parameters: $\theta_j = g_j\big(E(X), E(X^2), \ldots, E(X^m)\big)$.
- Replace population moments by **sample moments** $\overline{X^k} = \dfrac{1}{n}\sum_{i=1}^{n} X_i^k$.

  Since $E(\overline{X^k}) = E(X^k)$, for large n, $\overline{X^k} \approx E(X^k)$.
- The **method of moments estimator (MME)** is $\hat{\theta}_j = g_j\big(\overline{X}, \overline{X^2}, \ldots, \overline{X^m}\big)$.

Exercise

Let $X_1, \ldots, X_n$ be a random sample from the Gamma distribution $\Gamma(\alpha, \theta)$, so that the probability density function is given by

$$f(x; \alpha, \theta) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, \quad x > 0.$$

Knowing that $E(X_i) = \alpha\theta$ and $Var(X_i) = \alpha\theta^2$, find the method of moments estimator of α and θ.

Maximum Likelihood Estimator

Definition. For n iid samples $X_1, X_2, \ldots, X_n$ of a discrete rv X, the **maximum likelihood estimator (MLE)** of parameter(s) θ is

$$\hat{\theta} = \arg\max_{\theta} p_\theta(X_1, X_2, \ldots, X_n) = \arg\max_{\theta} \left(\prod_{i=1}^{n} p_\theta(X_i) \right).$$

Here $p_\theta(X_1, X_2, \ldots, X_n)$ and $p_\theta(X_i)$ are joint and marginal pmfs.

Exercise

Let $X_1, \ldots, X_n$ be samples from the uniform distribution $U(0, \theta)$. Find the MLE of θ.

Exercise

Let $X_1, \ldots, X_n$ be i.i.d. samples whose pdf is determined by another two-point parameter $\theta \in \{0, 1\}$. The pdf is

- $f(x; \theta = 0) = \mathbb{I}_{(0,1)}(x)$;
- $f(x; \theta = 1) = \dfrac{1}{2\sqrt{x}} \mathbb{I}_{(0,1)}(x)$.

Find the MLE of θ.

Exercise

Let $X_1, \ldots, X_n$ be i.i.d. samples from a population with the double exponential distribution with pdf

$$f(x;\theta) = \frac{1}{2}e^{-|x-\theta|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Find the MLE of θ.

Hint: Consider the cases of even and odd n separately.