MATH 2411 T1B Tutorial 7

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Interval Estimation

- We will provide an interval $[\hat{\theta} - a, \hat{\theta} + b]$ such that $C = P\big(\theta \in [\hat{\theta} - a, \hat{\theta} + b]\big)$ is large.
- $P\big(\theta \in [\hat{\theta} - a, \hat{\theta} + b]\big) = P(\hat{\theta} - a \leq \theta \leq \hat{\theta} + b) = P(\theta - b \leq \hat{\theta} \leq \theta + a)$.
- To answer this, we need to know the distribution of $\hat{\theta}$. This is a distribution across different sets of samples, so it is referred to as the sampling distribution.

Sampling Distribution of $\bar{X}$

Theorem (Central Limit Theorem): $X_1, \ldots, X_n$ are i.i.d. samples of r.v. X which has mean μ and variance σ^2, then we have the following limit in terms of distribution:

$$\lim_{n\to\infty} \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \to N(0, 1).$$

CI of $\bar{X}$ for estimating μ

CI of $\bar{X}$ for estimating μ at the confidence level C when σ^2 is known:

$$\left[\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}},\ \bar{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right]$$

Where $\alpha = (1 - C)/2$, $z_\alpha = \Phi^{-1}(1 - \alpha)$ is called the critical value.

R code

```
qnorm(1-(1-0.95)/2)
qnorm(1-(1-0.90)/2)
qnorm(1-(1-0.99)/2)
```

Theorem: For $X \sim N(\mu, \sigma^2)$ and with i.i.d. samples,

$$\frac{\bar{X} - \mu}{S_{n-1}/\sqrt{n}} \sim t_{n-1},$$

t_ν is the Student's t distribution with degree of freedom equal to ν.

Confidence Interval for μ with Unknown Variance

CI of μ when σ^2 is unknown:

$$\left[\bar{x} - t_{n-1,\alpha}\frac{s_{n-1}}{\sqrt{n}},\ \bar{x} + t_{n-1,\alpha}\frac{s_{n-1}}{\sqrt{n}}\right]$$

for confidence level $C = 1 - 2\alpha$.

R code examples for $t_{n-1,\alpha}$

```
qt(1-(1-0.95)/2, df=9)
qt(1-(1-0.90)/2, df=9)
qt(1-(1-0.99)/2, df=9)
```

Example

```
x = c(12, 23, 14, 18, 29, 30, 15, 20, 25, 16)
xbar = mean(x)
s = sd(x)
n = length(x)
C = 0.95
alpha = (1-C)/2
t = qt(1-alpha, df=n-1)
xbar + c(-1,1)*t*s/sqrt(n)
```

Or

```
t.test(x, alternative = "two.sided", conf.level=0.95)
```

Chi-squared Distribution

Theorem: For $X_1, \ldots, X_n$ are i.i.d. samples of $X \sim N(\mu, \sigma^2)$, then

$$\frac{(n-1)S_{n-1}^2}{\sigma^2} \sim \chi_{n-1}^2,$$

χ_k^2 is the chi-squared distribution with k degrees of freedom.

CI of σ^2

CI of σ^2 at the confidence level C:

$$\left[\frac{(n-1)s_{n-1}^2}{\chi_{n-1,\alpha}^2}, \frac{(n-1)s_{n-1}^2}{\chi_{n-1,1-\alpha}^2}\right]$$

Where $\alpha = (1-C)/2$, $\chi_{n-1,\alpha}^2 = \chi_{n-1}^2(1-\alpha)$ is called the critical value.

R code examples for $\chi^2_{n-1,\alpha}$

```
qchisq(1-(1-0.95)/2, df=119)
qchisq((1-0.95)/2, df=119)
qchisq((1-0.95)/2, df=119, lower.tail = F)
qchisq(1-(1-0.95)/2, df=119, lower.tail = F)
```

Example

```
x = c(12, 23, 14, 18, 29, 30, 15, 20, 25, 16,
      19, 22, 17, 21, 24, 27, 28, 26, 13, 11,
      10, 9, 8, 7, 6, 5, 4, 3, 2, 1)
s = sd(x)
n = length(x)
C = 0.95
alpha = (1-C)/2
chisq_1 = qchisq(1-alpha, df=n-1)
chisq_2 = qchisq(alpha, df=n-1)
c((n-1)*s^2/chisq_1, (n-1)*s^2/chisq_2)
```

Exercise

Derive the confidence interval for $1/\sigma$ at the confidence level C.

Exercise

The prices of a particular variety of rice, per kilogram, collected from 48 local stores in the suburban areas of a district vary with a mean of 3 dollars and a standard deviation of 1.6 dollars. Construct a 95% confidence interval for the mean price.

Exercise

Given a sample $X_1, \ldots, X_n$ from a normal population $N(\mu, 16)$, determine the minimum sample size n such that the confidence interval

$$[\bar{X} - 1,\ \bar{X} + 1]$$

has a confidence level of 0.90.

Exercise

The volume in a set of wine bottles is known to follow a $\mathcal{N}(\mu, 25)$ distribution. You take a sample of the bottles and measure their volumes. How many bottles do you have to sample to have a 95% confidence interval for μ with width no larger than 1?