MATH 3033 T2A Tutorial 1

Zhen HOU
zhouah@connect.ust.hk

Quick Review of Week 1

1 Euclidean Spaces

Definition(n-dimensional Euclidean space): $\mathbb{R}^n$ is the set of all n-tuples of real numbers, i.e. $\{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$, together with **addition** and **scalar multiplication** defined as follows:

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n),$$
$$\lambda(x_1, x_2, \cdots, x_n) = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n),$$

for any $(x_i, y_i, \cdots, x_n), (y_i, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Notation: Unless otherwise specified, we denote $x \in \mathbb{R}^n$ as $x = (x_1, x_2, \ldots, x_n)$. Similarly for $y, z, \cdots$.

Definition(Inner Product(dot product)): The **inner product** For any $x, y \in \mathbb{R}^n$, the **inner product** of x and y is defined as:

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n,$$

which satisfies **postive definiteness([I1])**, **symmetry([I2])**, and **linearity([I3])**.

Remark: 1. Generally, any vector space with an inner product(i.e. satisfying [I1], [I2], and [I3]) is called an **inner product space**. 2. In this sence, $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is a special inner product space.

Definition(Norm): For any $x \in \mathbb{R}^n$, the **norm** of x is defined as:

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

which satisfies **positive definiteness([N1])**, **absolute homogeneity([N2])**, and **triangle inequality([N3])**.

Remark: 1. Generally, any vector space with a norm(i.e. satisfying [N1], [N2], and [N3]) is called a **norm vector space**. 2. For any inner product space, the inner produce naturally induces a norm by $\|x\| = \sqrt{\langle x, x\rangle}$.

Definition(Standard Metric): For any $x, y \in \mathbb{R}^n$, the **standard metric** is defined as:

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},$$

which satisfies **Positivity([M1])**, **Symmetry([M2])**, and **Triangle Inequality([M3])**.

Remark: 1. Generally, any non-empty set with a metric(i.e. satisfying [M1], [M2], and [M3]) is called a **metric space**. 2. For any norm vector space, the norm naturally induces a metric by $d(x, y) = \|x-y\|$.

2 Open and Closed Sets

Definition: x is a **"Concept"** of a set A if **"Definition"** and all such points form a set called **"Set Name"** of A.

Concept	**Definition**	**Set Name**
Interior Point	$\exists r > 0$ s.t. $B(x,r) \subseteq A$.	**Interior** $\mathrm{Int}(A)$
Accumulation Point (or Limit Point)	$\forall r > 0, B^*(x,r) \cap A \neq \emptyset$. ($\iff \exists \{x_n\}_{n=1}^{\infty} \subseteq A$ s.t. $x_n \neq x$ and $x_n \to x$)	**Derived Set** A'
Isolated Point	$x \in A$ and x is not an accumulation point. ($\iff \exists r > 0$ s.t. $B(x,r) \cap A = \{x\}$)	
Adherent Point	$\forall r > 0, B(x,r) \cap A \neq \emptyset$. ($\iff \exists \{x_n\}_{n=1}^{\infty} \subseteq A$ s.t. $x_n \to x$)	**Closure** $\overline{A}$
Boundary Point	$\forall r > 0, B(x,r) \cap A \neq \emptyset$ and $B(x,r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$. ($\iff \exists \{x_n\}_{n=1}^{\infty} \subseteq A$ s.t. $x_n \to x$ and $\exists \{y_n\}_{n=1}^{\infty} \subseteq (\mathbb{R}^n \setminus A)$ s.t. $y_n \to x$)	**Boundary** ∂A

Remark: By definition, (1) $A \subseteq \overline{A}$, (2) $\partial A \subseteq \overline{A}$, (3) $\overline{A} = A \cup A'$.

Concept(open): A is open if any following conditions is satisfied:

(1) $A = \mathrm{Int}(A)$, i.e. $\forall x \in A, \exists r > 0$ s.t. $B(x,r) \subseteq A$.

(2) A is the union of a (possibly infinitely) collection of open sets.

(3) A is the intersection of a finite collection of open sets.

(4) $\mathbb{R}^n \setminus A$ is closed.

Concept(closed): A is closed if any following conditions is satisfied:

(1) $A = \overline{A}$.

(2) A is the intersection of a (possibly infinitely) collection of closed sets.

(3) A is the union of a finite collection of closed sets.

(4) $\mathbb{R}^n \setminus A$ is open.

(5) $\partial A \subseteq A$.

Theorem 2.5. Every non-empty open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

Remark:

(1) To show $A = B$, we need to show $A \subseteq B$ and $B \subseteq A$. To show $A \subseteq B$, we need to show that $\forall x \in A, x \in B$.

(2) To show A is countable, we need to find a injection from A to a countable set, such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^n$, etc.

Theorem 2.4. Let $A \subseteq \mathbb{R}^n$.

(a) $\partial A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$

(b) $\operatorname{Int}(A) = A \setminus \partial A$

(c) $\overline{A} = A \cup \partial A$

(d) Let $\mathcal{O} = \{X \mid X \subseteq A \text{ and } X \text{ is open}\}$. Then

$$\operatorname{Int}(A) = \bigcup_{X \in \mathcal{O}} X.$$

Hence $\operatorname{Int}(A)$ is the largest open set contained in A, i.e., $\operatorname{Int}(A) \subseteq \mathcal{O}$ and for any $X \in \mathcal{O}$, $X \subseteq \operatorname{Int}(A)$.

(e) Let $\mathcal{C} = \{Y \mid A \subseteq Y \text{ and } Y \text{ is closed}\}$. Then

$$\overline{A} = \bigcap_{Y \in \mathcal{C}} Y.$$

Hence $\overline{A}$ is the smallest closed set containing A, i.e., $\overline{A} \subseteq \mathcal{C}$ and for any $Y \in \mathcal{C}$, $\overline{A} \subseteq Y$.

Problem Set 1

1. (Parallelogram Law) Prove that for any vectors $u, v \in \mathbb{R}^n$, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

   Remark: This property can be generalized to any **inner product space**.

   Remark: An interesting question is: Is this property also true for any **norm vector space**? If not, for what kind of normed space is this property true?

2. (1) Prove that for any two sets A and B, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (2) Find such sets A and B that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

3. Assume $S \subseteq \mathbb{R}$. Prove that if S' is countable, then S is also countable.

4. Define the addition between two sets S_1 and S_2 as $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$.
   Assume that A_1 and A_2 are two non-empty subset of $\mathbb{R}$ and $A_1' \neq \emptyset$.
   Prove that $\overline{A}_1 + A_2' \subseteq (A_1 + A_2)'$.