MATH 3033 T2A Tutorial 10

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Problem Set 9

3. Write the Taylor series of functions below centered at $x = 0$ and then specify the interval of convergence.

   (2) $f(x) = \ln(1 - x - 2x^2)$.

4. Calculate $\sum_{n=1}^{\infty} \dfrac{n(n+1)}{(2n)!!}$.

Quick Review: Sequences and Series of Functions

3 Uniform Convergence and Differentiation

Theorem 3.1. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f, and for any $x \in [a,b]$,

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

4 Uniform Convergence and Integration

Definition 4.1.(Lower Riemann Sum and Upper Riemann Sum): Let $f : [a,b] \to \mathbb{R}$ be a bounded real-valued function. A partition P of $[a,b]$ is a finite set of points $t_0, t_1, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b$$

For $j = 1, 2, \ldots, n$, define

$$M_j = \sup f([t_{j-1}, t_j]), \quad m_j = \inf f([t_{j-1}, t_j]),$$

and define

$$L(f,P) = \sum_{j=1}^{n} m_j(t_j - t_{j-1}), \quad U(f,P) = \sum_{j=1}^{n} M_j(t_j - t_{j-1})$$

Then $L(f,P)$ is called the **lower Riemann sum** for f corresponding to P and $U(f,P)$ is the **upper Riemann sum** for f corresponding to P.

Clearly, we have $U(f,P) \geq L(f,P)$.

If we write $L(f) = \sup\{L(f,P) | P \text{ is a partition}\}$, $U(f) = \inf\{U(f,P) | P \text{ is a partition}\}$, then it can be shown that for any partition P and P' of $[a,b]$

$$L(f,P) \leq L(f) \leq U(f) \leq U(f,P').$$

Definition 4.2. $L(f)$ is called the **lower Riemann integral** of f on $[a,b]$; $U(f)$ is called the **upper Riemann integral** of f on $[a,b]$. If $L(f) = U(f)$, then f is said to be **Riemann integrable** on $[a,b]$, and the common value of $L(f)$ and $U(f)$ is written $\int_a^b f(x)\,dx$ or $\int_a^b f$, and is called the **Riemann integral** of f on $[a,b]$.

Theorem 4.1. If f is bounded on $[a,b]$, then f is Riemann integrable on $[a,b]$ if and only if, given any $\varepsilon > 0$, there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Theorem 4.2. Let $\{f_n\}$ be a sequence of Riemann integrable functions defined on $[a,b]$ such that it converges uniformly to f on $[a,b]$. Then f is also Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

Corollary 4.1. If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $\rho > 0$, then f is Riemann integrable on any closed subinterval of $(a-\rho, a+\rho)$. Moreover, for any $x \in (a-\rho, a+\rho)$, we have

$$\int_a^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

The power series on the right also has radius of convergence ρ.

5 Abel's Limit Theorem

Theorem 5.1. Suppose $\{f_n\}$ converges uniformly to f on U.

1. **(Subset Property)** If $V \subseteq U$, $\{f_n\}$ converges uniformly to f on V.
2. **(Union Property)** If $\{f_n\}$ also converges uniformly to f on W, then it converges uniformly to f on $U \cup W$.
3. **(Bounded Multiplier Property)** If $g(x)$ is a bounded function on U, $\{gf_n\}$ converges uniformly to gf on U.
4. **(Substitution Property)** If $h : U_0 \to U$, then $\{f_n \circ h\}$ converges uniformly to $f \circ h$ on U_0.

Theorem 5.2. (**Abel's limit theorem**) Suppose the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges pointwise to $S(x)$ on a closed and bounded interval $[u, v]$, then it converges uniformly to $S(x)$ on $[u, v]$. Moreover, we have

$$\lim_{x \to u^+} S(x) = S(u) = \sum_{k=0}^{\infty} a_k(u-c)^k \quad \text{and} \quad \lim_{x \to v^-} S(x) = S(v) = \sum_{k=0}^{\infty} a_k(v-c)^k$$

Remark: Note that for power series, in both **Theorem 2.4.**(To change the order of summation and differentiation) and **Corollary 4.1.**(To change the order of summation and integration), the conclusion holds for open interval $(a-\rho, a+\rho)$. But Abel's limit theorem can be applied to the closed interval $[u, v]$.

Problem Set 10

1. Calculate $\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x \in [-1, 1)$.

2. Calculate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

3. Write the Taylor series of functions below centered at $x = x_0$ and then specify the interval of convergence.

   (1) $f(x) = e^{x^2}$

   (2) $f(x) = \dfrac{x}{1 + x - 2x^2}$

4. Suppose a function f is integrable on every finite interval, and

$$\lim_{x \to +\infty} f(x) = l.$$

Prove that

$$\lim_{x \to +\infty} \frac{1}{x} \int_0^x f(t)\, dt = l.$$