MATH 3033 T2A Tutorial 11

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Problem Set 10

3. Write the Taylor series of functions below centered at $x = x_0$ and then specify the interval of convergence.

   (1) $f(x) = e^{x^2}$

   (2) $f(x) = \dfrac{x}{1 + x - 2x^2}$

4. Suppose a function f is integrable on every finite interval, and

$$\lim_{x \to +\infty} f(x) = l.$$

Prove that

$$\lim_{x \to +\infty} \frac{1}{x} \int_0^x f(t)\, dt = l.$$

Quick Review: Lebesgue Integration

1 Motivation

We want to define a more general kind of integrals such that

1. the integral of functions like Dirichlet function is well-defined, and
2. a sequence of the integrals of functions converges to the integral of the limit function under some convergence conditions that are less restrictive than uniform convergence.

Idea: The French mathematician **Henri Lebesgue** came up with the brilliant idea that instead of partitioning the interval $[a, b]$, we can partition the range of g.

2 Lebesgue Measure

Let $\mathcal{P}(\mathbb{R})$ be the set of all subsets of $\mathbb{R}$. We want to define a measure $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ which has the following desired properties:

1. **(Interval length)** $\mu(I) = l(I)$ for every interval $I \subseteq \mathbb{R}$.
2. **(Countably additive)** For every sequence $E_1, E_2, \ldots$ of pairwise disjoint subsets of $\mathbb{R}$,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

3. **(Translation invariant)** For any $c \in \mathbb{R}$ and any $E \subseteq \mathbb{R}$, $\mu(E + c) = \mu(E)$, where $E + c = \{x + c \in \mathbb{R} \mid x \in E\}$.

Remark 2.1. 3. Unfortunately, it is impossible to construct such a measure that satisfies all the above properties.

Definition 2.1.(Lebesgue outer measure) Suppose $S \subseteq \mathbb{R}$, the (**Lebesgue**) **outer measure** of S, denoted by $\mu^*(S)$, is defined by

$$\mu^*(S) = \inf\left\{\sum_{I \in \mathcal{C}} l(I) \;\middle|\; \mathcal{C} \text{ is a collection of countable open intervals that covers } S\right\}$$

Theorem 2.1. μ^* is translation invariant and $\mu^*(I) = l(I)$ for any interval $I \subseteq \mathbb{R}$.

Theorem 2.2.(Countable subadditivity) Let $\{S_n\}$ be a countable collection of sets in $\mathbb{R}$. Then

$$\mu^*\left(\bigcup_n S_n\right) \leq \sum_n \mu^*(S_n)$$

Then we can define **a class of sets** in $\mathbb{R}$ such that μ^* is countable additive. The following definition is due to Carathéodory and is sometimes called the **Carathéodory's criterion:**

Definition 2.2. A set $E \subseteq \mathbb{R}$ is said to be **measurable** if for each set $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

where $E^c = \mathbb{R} \setminus E$ is the complement of E. The set of all measurable sets is denoted by $\mathfrak{M}$.

Remark The output measure is not countable additive on all subsets of $\mathbb{R}$. So we need to restrict the domain of the measure to the set of all measurable sets.

Remarks 2.2.

1. Since we always have $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$, E is measurable if and only if for each $A \subseteq \mathbb{R}$, we have
$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

2. As the definition of measurability is symmetric in E and E^c, E^c is measurable whenever E is.

3. Obviously, $\emptyset$ and $\mathbb{R}$ are measurable.

Theorem 2.3. If $\mu^*(E) = 0$, then E is measurable and is said to have **Lebesgue measure zero**.

Remarks 2.3.

1. Obviously, any singleton in $\mathbb{R}$ has measure zero. Then by countable subadditivity of μ^*, any countable set in $\mathbb{R}$ also has measure zero.

2. By countable subadditivity of μ^* again, any union of countable collection of sets of measure zero has measure zero.

Definition 2.3. Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$. We say that $\mathcal{C}$ is a **σ-algebra** if it satisfies the following conditions:

- $\emptyset \in \mathcal{C}$
- If $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$.
- If $\{A_n\}$ is a countable collection of sets in $\mathcal{C}$, then $\bigcup_n A_n \in \mathcal{C}$.

Remarks 2.4. The above definition implies the following:

- $\mathbb{R} \in \mathcal{C}$.
- If $\{A_n\}$ is a countable collection of sets in $\mathcal{C}$, then $\bigcap_n A_n \in \mathcal{C}$.

Theorem 2.4. $\mathfrak{M}$ is a σ-algebra.

Theorem 2.5. Let μ be the outer measure μ^* restricted on the σ-algebra $\mathfrak{M}$. Then $\mu : \mathfrak{M} \to [0, \infty]$ is a measure satisfying the three desired properties:

1. $\mu(I) = l(I)$ for any interval $I \subseteq \mathbb{R}$.

2. μ is countably additive.

3. μ is translation invariant.

The measure μ on $\mathbb{R}$ is called the **Lebesgue measure**.

Lemma 2.1. For any $a \in \mathbb{R}$, (a, ∞) is measurable.

Theorem 2.6. All open sets and closed sets in $\mathbb{R}$ are measurable.

Remarks 2.5. Let $\mathcal{B}$ be the collection of **Borel sets**, the smallest σ-algebra containing all open sets of $\mathbb{R}$. Then by the above theorem, $\mathcal{B} \subseteq \mathfrak{M}$.

4 Measurable Functions

Definition 4.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **measurable** if $f^{-1}(U)$ is measurable for any open set $U \subseteq \mathbb{R}$.

Example 4.1. (How big is the class of measurable functions?)

1. All continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are measurable.
2. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ such that f is measurable and g is continuous. Then $g \circ f$ is measurable.
3. The set E in Example 2.1 is a non-measurable sets. Therefore, it can be shown that χ_E is a non-measurable function.

Theorem 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The following statements are equivalent:

1. f is measurable.
2. $f^{-1}((a, \infty))$ is measurable for all $a \in \mathbb{R}$.
3. $f^{-1}([a, \infty))$ is measurable for all $a \in \mathbb{R}$.
4. $f^{-1}((-\infty, a))$ is measurable for all $a \in \mathbb{R}$.
5. $f^{-1}((-\infty, a])$ is measurable for all $a \in \mathbb{R}$.

Theorem 4.2. Suppose $c \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are measurable functions. $f + c$, cf, $f \pm g$, fg, $\max\{f, g\}$, $\min\{f, g\}$, and $|f|$ are measurable.

Theorem 4.3. If f is a measurable function and $f = g$ a.e., then g is also measurable.

Theorem 4.4. Let $A \subseteq \mathbb{R}$. Then χ_A is a measurable function if and only if A is measurable.

Definition 4.2. A function $f : \mathbb{R} \to \mathbb{R}$ is called a **simple function** if there exists finitely many measurable sets $A_1, A_2, \ldots, A_n \subseteq \mathbb{R}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}$$

Remarks 4.1.

1. Any simple function is a measurable function.

2. If f, g are simple functions, then $f+g, f-g, fg, \max\{f,g\}, \min\{f,g\}$, and $|f|$ are also simple.

Theorem 4.5. For any non-negative measureable function $f : \mathbb{R} \to [0,\infty)$, there exists a monotonically increasing sequence $\{s_n\}$ of simple functions i.e. $0 \leq s_1 \leq s_2 \leq \dots$ such that $\{s_n\}$ converges pointwise to f (Notation: $s_n \nearrow f$ on $\mathbb{R}$).

5 Lebesgue Integrals

First we define the Lebesgue integrals for non-negative simple functions:

Definition 5.1.(Lebesgue integral for non-negative simple functions) Let $s = \sum_{i=1}^n a_i \chi_{A_i}$ be a non-negative simple function i.e. $a_i \geq 0$ for all i. Then the Lebesgue integral of s is

$$\int s \, d\mu = \sum_{i=1}^n a_i \mu(A_i)$$

Moreover, if $E \subseteq \mathbb{R}$ is measurable, the Lebesgue integral of s over E is

$$\int_E s \, d\mu = \int s \chi_E \, d\mu = \sum_{i=1}^n a_i \mu(A_i \cap E)$$

(Convention: If $a_i \mu(A_i \cap E)$ is of the form $0 \cdot \infty$, we take this to be 0.)

Then we define the Lebesgue integrals for non-negative measurable functions:

Definition 5.2. Let $f : \mathbb{R} \to [0,\infty)$ be a non-negative measurable function and let $E \subseteq \mathbb{R}$ be a measurable set. Then the **Lebesgue integral** of f on E is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \;\middle|\; 0 \leq s \leq f \text{ and } s \text{ is a simple function} \right\}$$

Moreover, if $\int_E f \, d\mu$ is finite, f is said to be **Lebesgue integrable** over E.

Theorem 5.1. (Fatou's Lemma) Suppose $\{f_n\}$ is a sequence of non-negative measurable functions, then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Theorem 5.2. (Monotone Convergence Theorem MCT) Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions which converges a.e. to f. Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Theorem 5.3. Suppose $f, g : \mathbb{R} \to [0,\infty)$ are non-negative measurable functions and $a, b > 0$. Then

1. $\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu$;
2. $\int f \, d\mu = 0$ if and only if $f = 0$ a.e..

Theorem 5.4. If $f_n : \mathbb{R} \to [0, \infty)$ is a measurable function for all $n \in \mathbb{N}$, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Definition 5.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and $E \in \mathfrak{M}$. We say that f is **Lebesgue integrable** over E if $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are Lebesgue integrable over E i.e. $\int_E f^+ \, d\mu$ and $\int_E f^- \, d\mu$ are both finite. Then we define

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu.$$

Theorem 5.5. (Lebesgue's Dominated Convergence Theorem LDCT) Let $E \in \mathfrak{M}$. Suppose $g : E \to [0, \infty]$ is Lebesgue integrable on E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n(x)| \leq |g(x)|$ $(n \in \mathbb{N}, x \in E)$ and $f_n \to f$ a.e. on E. Then f is Lebesgue integrable on E and

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu.$$

Theorem 5.6. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded real-valued function such that it is Riemann integrable on $[a, b]$ i.e. $\int_a^b f(x) \, dx$ exists and is finite. Then f is Lebesgue integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, d\mu.$$

Example 5.1. is an example of a real-valued continuous function on $[0, \infty)$ such that the improper Riemann integral $\int_0^\infty f(x) \, dx$ converges but f is NOT Lebesgue integrable on $[0, \infty)$.

Theorem 5.7. Suppose $f : [0, \infty) \to [0, \infty)$ is a nonnegative measurable function. Then the improper Riemann integral $\int_0^\infty f(x) \, dx$ converges implies that f is Lebesgue integrable and

$$\int_{[0,\infty)} f \, d\mu = \int_0^\infty f(x) \, dx.$$

Theorem 5.8. Let I be an interval (possibly unbounded) and $f \in L_{\text{loc}}(I)$ i.e. f is Riemann integrable on all $[s, t] \subseteq I$. If $|f|$ is improper Riemann integrable on I, then f is improper Riemann integrable and Lebesgue integrable on I and two integrals give the same value.

Problem Set 11

1. Prove that if $A \in \mathfrak{M}$, then for any $B \subseteq \mathbb{R}$,

$$\mu^*(A \cap B) + \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

2. Let $E \subseteq \mathbb{R}$. Prove that if $\forall \epsilon > 0$, $\exists A(\epsilon), B(\epsilon) \in \mathfrak{M}$ satisfying $A(\epsilon) \subseteq E \subseteq B(\epsilon)$ and $\mu^*(B(\epsilon) \setminus A(\epsilon)) < \epsilon$, then $E \in \mathfrak{M}$.

3. Find such open set $A \subseteq \mathbb{R}$ that $\mu(A) \neq \mu(\overline{A})$.

4. Let $f(x)$ be a monotonic function. Prove that $f(x)$ is measurable.

5. Let $f_k(x) : [a,b] \to \mathbb{R}$ be measurable functions for $k = 1, 2, \cdots$. Prove that $\exists \{c_k\} \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} c_k \cdot f_k(x) \text{ exists and is finite a.e. } x \in [a,b].$$

6. Find an example to show that the inequality in the Fatou's lemma can be strict.

7. Denote $[0,1] \cap \mathbb{Q}$ by $\{r_k\}_{k=1}^{\infty}$. Let $f(x) = \sum_{k=1}^{\infty} \dfrac{1}{k^2 \cdot |x - r_k|^{1/2}}$. Prove that $f(x) < \infty$, a.e. $x \in [0,1]$. **Note:** $f(x)$ is discontinuous everywhere and unbounded on every interval in $[0,1]$.

8. Let f be Lebesgue integrable over $[0, \infty)$. Prove that $\lim_{n\to\infty} f(x+n) = 0$ a.e. $x \in [0, \infty)$.