MATH 3033 T2A Tutorial 2

Zhen HOU
zhouah@connect.ust.hk

Quick Review: Elemens of Point-Set Topology

3 Compactness

Definition(Covering and Open Covering): A collection $\mathcal{G}$ of sets in $\mathbb{R}^n$ is called a **covering** of a given set S (or $\mathcal{G}$ covers S) if $S \subseteq \bigcup_{A \in \mathcal{G}} A$. Moreover, if each member of $\mathcal{G}$ is an open set, then we say $\mathcal{G}$ is an **open covering**.

Definition(Subcovering): Given a set $S \subseteq \mathbb{R}^n$. Let $\mathcal{G}$ is a covering of S and $\mathcal{D} \subseteq \mathcal{G}$ such that $\mathcal{D}$ also covers S, then we say $\mathcal{D}$ is a **subcovering** of $\mathcal{G}$.

Definition(Compact): Let $S \subseteq \mathbb{R}^n$. Then S is said to be **compact** if any open covering of S has a finite subcovering.

Definition(Bounded): Let $S \subseteq \mathbb{R}^n$. Then S is said to be **bounded** if there exists $a \in \mathbb{R}^n$ and $r > 0$ such that $S \subseteq B(a, r)$.

Theorem 3.1. Let $S \subseteq \mathbb{R}^n$($\mathbb{R}^n$ can be replaced by any metric space). Then

(a) If S is compact, then S is closed and bounded.

(b) S is compact if and only if for any infinite subset T of $S, T' \cap S \neq \emptyset$.

Remark: For a set $S \subseteq \mathbb{R}^n$, The following two are always open covering:

1. Fix $a \in \mathbb{R}^n$, $\mathcal{G} = \{B(a, r) \mid r > 0\}$.
2. $\mathcal{F} = \{B(x, r_x) \mid x \in S, r_x > 0\}$. (How to choose r_x?)

4 Sequences of Points in $\mathbb{R}^n$

Definition(Convergence): Let $\{x_m\}$ be a sequence of points in $\mathbb{R}^n$. A point $l \in \mathbb{R}^n$ is the limit of $\{x_m\}$ if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - l\| < \epsilon$ for all $m > N$, and we write

$$\lim_{m \to \infty} x_m = l$$

In other words, $x_m \in B(l, \epsilon)$ for all $m > N$. We say that the sequence $\{x_m\}$ **converges** if $\lim_{m \to \infty} x_m = l$ for some $l \in \mathbb{R}^n$.

Definition(Sequentially Compact): A set $S \subseteq \mathbb{R}^n$ is said to be **sequentially compact** if for any sequence of points in S, there exists a subsequence converging to a point in S.

Remark: Theorem 3.1(b) in the previous section can be rephrased as follows: S is compact if and only if S is sequentially compact.

Definition(Cauchy Sequence): A sequence of points $\{x_m\}$ in $\mathbb{R}^n$ is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - x_{m'}\| < \epsilon$ for all $m, m' > N$.

Theorem 4.1. Let $\{x_m\}$ be a sequence in $\mathbb{R}^n$ such that for each $m \in \mathbb{N}, x_m$ has coordinates $(x_{1m}, x_{2m}, \ldots, x_{nm})$. If $l \in \mathbb{R}^n$ with coordinates $(l_1, l_2, \ldots, l_n)$, then

$$\lim_{m\to\infty} x_m = l \Leftrightarrow \lim_{m\to\infty} x_{km} = l_k \text{ for } k = 1, 2, \ldots, n$$

Theorem 4.2. Suppose $\{x_m\}$ is a sequence in $\mathbb{R}^n$. Then $\{x_m\}$ converges if and only if $\{x_m\}$ is a Cauchy sequence.

Theorem 4.3. (Bolzano-Weierstrass Theorem) Let T be an infinite and bounded subset of $\mathbb{R}^n$. Then $T' \neq \emptyset$.

Theorem 4.4. (Heine-Borel Theorem) Let S be a closed and bounded subset of $\mathbb{R}^n$. Then S is compact.

Remark:

1. Let $S \subseteq \mathbb{R}^n$. S is compact if and only if S is closed and bounded.
2. Let S be a subset of metric space. If S is compact then S is closed and bounded.
3. Let S be a subset of metric space. S is compact if and only if S is sequentially compact.

Remark:

1. Choice of radius.
2. Choice of point in a nonempty set.

Problem Set 1

3. Assume $S \subseteq \mathbb{R}$. Prove that if S' is countable, then S is also countable.

4. Define the addition between two sets S_1 and S_2 as $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$.
   Assume that A_1 and A_2 are two non-empty subset of $\mathbb{R}$ and $A_1' \neq \emptyset$.
   Prove that $\overline{A}_1 + A_2' \subseteq (A_1 + A_2)'$.

Problem Set 2

Notation: In the following, denote A° as $\text{Int}(A)$ and A^c as $\mathbb{R}^n \backslash A$ for simplicity.

2. Show $A^\circ = \left(\overline{A^c}\right)^c$, thus $\partial A = \bar{A} \cap (A^\circ)^c$.

3. Let $E \subseteq \mathbb{R}^n$, show ∂E is a closed set.

   Hint: Show $(\partial E)^c = E^\circ \cup (E^c)^\circ$.

4. For $E \subseteq \mathbb{R}^n$, show $\partial \bar{E} \subseteq \partial E$.

   Hint: use $\partial E = \bar{E} \cap \overline{E^c}$ for E and $\bar{E}$.

5. Prove that $A \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ For a collection of closed sets $\mathcal{F} = \{A_\alpha\}$, such that $A \cap (\cap_\alpha A_\alpha) = \emptyset$, we can find $\{A_1, \cdots, A_k\} \subseteq \mathcal{F}$ such that $A \cap \left(\cap_{i=1}^k A_i\right) = \emptyset$.

   Hint: Use definition of compact set and $(\cup A_\alpha)^c = \cap A_\alpha^c$.

6. (1) Let $F_1, F_2, F_3, \ldots$ are non-empty compact sets in $\mathbb{R}^n$ satisfying $F_{k+1} \subset F_k (k \geqslant 1)$. Prove that $\cap_{k=1}^{\infty} F_k \neq \emptyset$.

   Remark: In additionally if $\operatorname{diam}(F_k) \to 0$, then the intersection is a singleton.

   (2) Find nonempty closed sets $F_1, F_2, F_3, \ldots$ in $\mathbb{R}^n$ satisfying $F_{k+1} \subset F_k (k \geqslant 1)$ such that $\cap_{k=1}^{\infty} F_k = \emptyset$.