MATH 3033 T2A Tutorial 3

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Problem Set 2

Notation: In the following, denote A° as $\text{Int}(A)$ and A^c as $\mathbb{R}^n \backslash A$ for simplicity.

4. For $E \subseteq \mathbb{R}^n$, show $\partial \bar{E} \subseteq \partial E$.

   Hint: use $\partial E = \bar{E} \cap \overline{E^c}$ for E and $\bar{E}$.

5. Prove that $A \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ For a collection of closed sets $\mathcal{F} = \{A_\alpha\}$, such that $A \cap (\cap_\alpha A_\alpha) = \emptyset$, we can find $\{A_1, \cdots, A_k\} \subseteq \mathcal{F}$ such that $A \cap \left(\cap_{i=1}^k A_i\right) = \emptyset$.

   Hint: Use definition of compact set and $(\cap A_\alpha)^c = \cup A_\alpha^c$.

6. (1) Let $F_1, F_2, F_3, \ldots$ are non-empty compact sets in $\mathbb{R}^n$ satisfying $F_{k+1} \subset F_k (k \geqslant 1)$. Prove that $\cap_{k=1}^{\infty} F_k \neq \emptyset$.

   Remark: In additionally if $\operatorname{diam}(F_k) \to 0$, then the intersection is a singleton.

   (2) Find nonempty closed sets $F_1, F_2, F_3, \ldots$ in $\mathbb{R}^n$ satisfying $F_{k+1} \subset F_k (k \geqslant 1)$ such that $\cap_{k=1}^{\infty} F_k = \emptyset$.

Quick Review: Continuity and Differentiability

1 Continuity

Definition(Limit and Continuity): Suppose $U \subseteq \mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ is a (vector-valued) function. Let $a \in U'$ and $l \in \mathbb{R}^n$. We say that the **limit** of $f(x)$ as x approaches a exists and equals l if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $||f(x) - l|| < \varepsilon$ for all $x \in U$ satisfying $0 < ||x - a|| < \delta$, denoted by

$$\lim_{x \to a} f(x) = l$$

Moreover, if $a \in U$ and $\lim\limits_{x \to a} f(x) = f(a)$, we say that f is **continuous** at a. f is continuous on U if f is continuous at every point in U.

Remark(Rewritten with Topology): The above definition of limit can be rewritten as follows: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B^*(a, \delta) \cap U) \subseteq B(l, \varepsilon)$$

Remark: In the definition of limit, we require $a \in U'$, not $a \in U$. Note that generally, $U' \nsubseteq U$ and $U \nsubseteq U'$.

Theorem(Equivalence with limit of coordinate functions): Let $f : U \to \mathbb{R}^n$ is a function, where $U \subseteq \mathbb{R}^m$. Let $a \in U'$ and $l = (l_1, l_2, ..., l_n) \in \mathbb{R}^n$. Then

$$\lim_{x \to a} f(x) = l \iff \lim_{x \to a} f_i(x) = l_i \text{ for } i = 1, 2, ..., n$$

where f_i is the i^{th} coordinate function of f. Moreover, if $a \in U$, then f is continuous at a if and only if f_i is continuous at a for $i = 1, 2, ..., n$.

Theorem(Generalization of Sequential Limit Theorem): Suppose $U \subseteq \mathbb{R}^m$ and $a \in U$. Let $f : U \to \mathbb{R}^n$. Then $\lim\limits_{x \to a} f(x) = L$, where $L \in \mathbb{R}^n$ if and only if for any sequence $\{x_k\}$ in $U \setminus \{a\}$ converging to a i.e. $\lim\limits_{k \to \infty} x_k = a$, $\lim\limits_{k \to \infty} f(x_k) = L$.

Remark: From this theorem, if we want to show that a function is not continuous at a point, we can find a sequence that converges to the point but the sequence of images does not converge to the image of the point, i.e. to find a sequence $\{x_k\}$ in $U \setminus \{a\}$ converging to a such that $\lim\limits_{k \to \infty} f(x_k) \neq f(a)$.

Theorem(Continuity of Sum, Product, and Quotient of Continuous Functions): Let $U \subseteq \mathbb{R}^m$ and $f, g : U \to \mathbb{R}$ are continuous at $a \in U$. Then we have

(a) $f + g$ is continuous at a.

(b) fg is continuous at a.

(c) $\dfrac{f}{g}$ is continuous at a if $g(a) \neq 0$.

Theorem(Inverse Image of Open Set is Open in U): Let $U \subseteq \mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is continuous on U if and only if the inverse image of every open set under f is **open in** U i.e. for every open set O in $\mathbb{R}^m$, $f^{-1}(O) = W \cap U$ for some open set W in $\mathbb{R}^n$.

Remark: The above theorem (1) still holds if we replace $\mathbb{R}$ with any metric space; (2) can be use as a definition of continuitous function between topological spaces.

Corollary(Inverse Image of Closed Set is Closed in U): Suppose $U \subseteq \mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$ if and only if the inverse image of every closed set under f is **closed in** U i.e. for every closed set C in $\mathbb{R}^m$, $f^{-1}(C) = V \cap U$ for some closed set V in $\mathbb{R}^n$.

Corollary(Continuity of Composition of Continuous Functions): Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$. If $f : U \to V$ and $g : V \to \mathbb{R}^k$ are continuous, then $g \circ f : U \to \mathbb{R}^k$ is also continuous. (Then the contuity of sum, product, and quotient of continuous functions can be further corollary.)

Theorem(Continuous Function can Preserve Compactness): Suppose $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^m$ is continuous on K, then $f(K)$ is compact in $\mathbb{R}^m$.

Corollary(Continuous Function on Compact Set attains Maximum and Minimum): Suppose $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous on K. Then f attains its maximum (minimum) value on K i.e. there exists $x_0 \in K$ such that $f(x_0) \geq f(x)$ $(\leq f(x))$ for all $x \in K$.

Remark: Recall that in $\mathbb{R}^n$, compact is equivalent to closed and bounded.

Definition(Lipschitz Continuous): Suppose $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$. f is said to be **Lipschitz continuous** on U if there exists a constant $C \geq 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\| \text{ for any } x, y \in U$$

Moreover, if $n = m$ and $0 \leq C < 1$, f is called a **contraction**.

Suppose U is open and $a \in U$. f is said to be **Lipschitz continuous at** a if there exists $\delta > 0$ and a constant $C \geq 0$ such that for any $x \in B(a, \delta)$, $\|f(x) - f(a)\| \leq C\|x - a\|$.

Theorem(Lipschitz Continuity implies Continuity): Suppose $U \subseteq \mathbb{R}^n$ is open, $a \in U$ and $f : U \to \mathbb{R}^m$ is Lipschitz continuous on a. Then f is continuous on a. Moreover, if f is Lipschitz continuous on U, then f is continuous on U.

Summary: Suppose $U \subseteq \mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ is a function. Let $a \in U'$.

1. To check $\lim\limits_{x \to a} f(x) = l$, use definition of limit or one of its equivalent statements.
2. To check $\lim\limits_{x \to a} f(x)$ does not exist,
   (1) Find $\{x_k\}$ in $U \setminus \{a\}$ s.t. $\lim\limits_{k \to \infty} x_k = a$ but $\lim\limits_{k \to \infty} f(x_k)$ does not exist.
      (i) Find $\{x_k\}$ in $U \setminus \{a\}$ s.t. $\lim\limits_{k \to \infty} x_k = a$ but $f(x_k)$ is unbounded.
      (ii) Find two sequences $\{x_k\}$ and $\{b_k\}$ in $U \setminus \{a\}$ s.t. $\lim\limits_{k \to \infty} x_k = a$ and $\lim\limits_{k \to \infty} b_k = a$ but $\lim\limits_{k \to \infty} f(x_k) \neq \lim\limits_{k \to \infty} f(b_k)$.

   Moreover if $a \in U$,
3. To check f is continuous at a,
   (1) check f is Lipschitz continuous at a.
   (2) use definition of continuity at a.
4. To check f is not continuous at a,
   (1) check $\lim\limits_{x \to a} f(x)$ does not exist.
   (2) check $\lim\limits_{x \to a} f(x) \neq f(a)$.
5. To check f is continuous on U,
   (1) check f is Lipschitz continuous on U.
   (2) use Corollary 1.3., i.e. write f as a composition of continuous functions.
   (3) check f is continuous at every point in U.
   (4) check inverse image of every open set under f is open in U.
   (5) check inverse image of every closed set under f is closed in U.

Problem Set 3

1. Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^2y^2}{|x|^3 + y^6}.$$

   Show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x \sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

(1) Prove that f is continuous at $(0,0)$.

(2) Prove that f is not continuous at $(x_0, 0)$ for any $x_0 \neq 0$.

3. Let $F \subset \mathbb{R}$ be closed. Prove that there exists $E \subset F$, such that E is countable and $\bar{E} = F$.

   Hint: 1. $\mathbb{Q}$ is countable. 2. If A and B are both countable, then Cartesian product $\{(x, y)|x \in A, y \in B\}$ is countable.

4. Let $F \subset \mathbb{R}$ be closed. Try to design $f : \mathbb{R} \to \mathbb{R}$, s.t. $F = \{a \in \mathbb{R} | f \text{ is not continuous at } a\}$.

   Hint: Use the former question.