MATH 3033 T2A Tutorial 4

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Problem Set 3

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x\sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

   (1) Prove that f is continuous at $(0,0)$.

   (2) Prove that f is not continuous at $(x_0, 0)$ for any $x_0 \neq 0$.

3. Let $F \subset \mathbb{R}$ be closed. Prove that there exists $E \subset F$, such that E is countable and $\bar{E} = F$.

   Hint: 1. $\mathbb{Q}$ is countable. 2. If A and B are both countable, then Cartesian product $\{(x, y) | x \in A, y \in B\}$ is countable.

4. Let $F \subset \mathbb{R}$ be closed. Try to design $f : \mathbb{R} \to \mathbb{R}$, s.t. $F = \{a \in \mathbb{R} | f \text{ is not continuous at } a\}$.
   Hint: Use the former question.

Quick Review: Continuity and Differentiability

2 Differentiability

Definition(Differentiability): Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$. then we say f is **differentiable** at a if there exists $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0$$

and in this case L is called the **derivative** of f at a and is denoted by $Df(a)$.

Theorem(Uniqueness of Derivative): The linear transformation L in the above definition is unique i.e. the definition of the derivative of f is well-defined.

Lemma(Lipschitz Continuity of Linear Transformation): Suppose $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then L is Lipschitz continuous on $\mathbb{R}^n$.

Remark 2.1.:

1. $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ can be regarded as a normed vector space with norm $\|L\| = \sqrt{\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij}^2\right)}$ for any $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then, by the above lemma, we have $\|L(h)\| \leq \|L\|\|h\|$ for any $h \in \mathbb{R}^n$.

   Note: (1) For matrix sense, this norm is also known as the Frobenius norm. (2) Viewing $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as a vector space, this norm is $\mathbb{R}^{mn}$-norm with a specific choice of basis.

2. We can also define another norm $\|L\|_{op} = \max_{\|h\|=1} \|L(h)\|$ on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. It is called the **operator norm** of L. In other words, $\|L\|_{op}$ is the "best constant" for the inequality $\|L(h)\| \leq C\|h\|$ i.e. the smallest possible C is $\|L\|_{op}$. By this lemma, $\|L\|_{op} \leq \|L\|$. In fact, it can be shown that
$$\|L\|_{op} \leq \|L\| \leq \sqrt{n}\|L\|_{op}$$

Theorem 2.2. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be differentiable at $a \in U$. Then f is Lipschitz continuous at a. In particular, f is continuous at a (by Theorem 1.6).

Theorem(Chain rule): Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are functions such that f is differentiable at $a \in \mathbb{R}^n$ and g is differentiable at $f(a) \in \mathbb{R}^m$. Then $g \circ f$ is differentiable at a and

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

Theorem 2.4.

(a) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a constant, then $Df(a) = 0$ for any $a \in \mathbb{R}^n$.

(b) If $f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then f is differentiable on $\mathbb{R}^n$ and $Df(a) = f$ for any $a \in \mathbb{R}^n$.

(c) Suppose $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$ is a function such that $f = (f_1, f_2, \ldots, f_m)$, where $f_i : U \to \mathbb{R}$ for $i = 1, 2, \ldots, m$. Then f is differentiable at $a \in U$ if and only if f_i is differentiable at a. Moreover, $Df(a) = (Df_1(a), Df_2(a), \ldots, Df_m(a))$.

(d) If $s : \mathbb{R}^2 \to \mathbb{R}$ is defined by $s(x, y) = x + y$, then $Ds(a_1, a_2) = s$ for any $(a_1, a_2) \in \mathbb{R}^2$.

(e) If $p : \mathbb{R}^2 \to \mathbb{R}$ is defined by $p(x, y) = xy$, then $Dp(a_1, a_2)(h_1, h_2) = a_2h_1 + a_1h_2$ for any $(a_1, a_2) \in \mathbb{R}^2$.

Corollary 2.1. Suppose $U \subseteq \mathbb{R}^n$ is open and $f, g : U \to \mathbb{R}$ are differentiable at a. Then

(a) $f + g$ is differentiable at a and $D(f + g)(a) = Df(a) + Dg(a)$.

(b) (Product rule) fg is differentiable at a and $D(fg)(a) = g(a)Df(a) + f(a)Dg(a)$.

(c) (Quotient rule) If $g(a) \neq 0$, $\frac{f}{g}$ is differentiable at a and

$$D\left(\frac{f}{g}\right) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}$$

3 Jacobian Matrix

Definition(Directional Derivative): Let $U \subseteq \mathbb{R}^n$ be open and $a \in U$. Suppose $f : U \to \mathbb{R}^m$. For each vector $v \in \mathbb{R}^n$, the **directional derivative** of f along v at a is

$$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}$$

In particular, if $\{e_1, e_2, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, then the directional derivative of f along e_j at a is the **partial derivative** of f with respect to the j^{th} variable at a, denoted by $\frac{\partial f}{\partial x_j}(a)$, $D_jf(a)$, or $\partial_jf(a)$.

Theorem 3.1. Let $U \subseteq \mathbb{R}^n$ be open, $a \in U$ and $f : U \to \mathbb{R}^m$. If f is differentiable at a, then for each vector $v \in \mathbb{R}^n$, the directional derivative of f along v at a exists and equals $Df(a)(v)$.

The following theorem gives a sufficient condition for a function to be differentiable:

Theorem Let $U \subseteq \mathbb{R}^n$, $a \in U$ and $f = (f_1, f_2, \ldots, f_m) : U \to \mathbb{R}^m$. If $\partial_jf_i(x)$ exists for all $x \in B(a, r) \subseteq U$ and is continuous at a for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then f is differentiable at a. Such function is said to be continuously differentiable at a.

Theorem 3.3. Let $U \subseteq \mathbb{R}^n$ be open and $\mathbf{f} = (f_1, f_2, \ldots, f_m) : U \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in U$. For any $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$,

$$D\mathbf{f}(\mathbf{a})\mathbf{h} = \begin{pmatrix} \partial_1f_1(\mathbf{a}) & \partial_2f_1(\mathbf{a}) & \cdots & \partial_nf_1(\mathbf{a}) \\ \partial_1f_2(\mathbf{a}) & \partial_2f_2(\mathbf{a}) & \cdots & \partial_nf_2(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1f_m(\mathbf{a}) & \partial_2f_m(\mathbf{a}) & \cdots & \partial_nf_m(\mathbf{a}) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

The $m \times n$ matrix of $D\mathbf{f}(\mathbf{a})$ is called the **Jacobian matrix** of $\mathbf{f}$.

Summary: Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$.

1. To prove f is differentiable at a, use the definition. Note that we can first try the jacobian matrix as the L in the definition(Theorem 3.3).
2. To prove f is not differentiable at a,
   (1) try to prove f is not continuous at a(Theorem 2.2).
   (2) try to prove some directional derivative of f does not exist at a(Theorem 3.1).
   (3) try to prove the limit in the definition of differentiability does not exist with L as the Jacobian matrix(Theorem 3.3).
3. Note that the problem about directional derivative is basically a problem about sequences.

Problem Set 4

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = \sqrt{|xy|}$. Try to prove:
   (a) f is continuous at $(0, 0)$.
   (b) Both $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist and are equal to 0.
   (c) f is not differentiable at $(0, 0)$.

2. Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & x^2+y^2>0, \\ 0, & x=y=0. \end{cases}$$

Prove the function f has directional derivatives in every direction at the origin, but f is not differentiable at the origin.

3. Let

$$f(x,y) = \begin{cases} xy\sin\left(\frac{1}{x^2+y^2}\right), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show the function is differentiable at $(0,0)$, but its two partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are not continuous at $(0,0)$.

4. Compute the Jacobian matrix and the differential of the following maps at the specified points:

   (a) $f(x, y) = (xy^2 - 3x^2, 3x - 5y^2)$, at $(1, -1)$;
   (b) $f(x, y, z) = (xy^2 - 4y^2, 3xy^2 - y^2 z)$, at $(1, -2, 3)$;
   (c) $f(x, y) = (e^x \cos(xy), e^x \sin(xy))$, at $(1, \pi/2)$.

5. Let $u(x, y, z) = f(x^2 + y^2)$. Prove that

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

6. Let $f(x, y, z) = F(u, v, w)$, where $x^2 = vw$, $y^2 = wu$, $z^2 = uv$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}.$$