MATH 3033 T2A Tutorial 5

Zhen HOU
zhouah@connect.ust.hk

Problem Set 4

2. Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & x^2 + y^2 > 0, \\ 0, & x = y = 0. \end{cases}$$

Prove the function f has directional derivatives in every direction at the origin, but f is not differentiable at the origin.

3. Let

$$f(x,y) = \begin{cases} xy \sin\left(\dfrac{1}{x^2+y^2}\right), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show the function is differentiable at $(0,0)$, but its two partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are not continuous at $(0,0)$.

4. Compute the Jacobian matrix of the following maps at the specified points:

   (a) $f(x, y) = (xy^2 - 3x^2, 3x - 5y^2)$, at $(1, -1)$;

   (b) $f(x, y, z) = (xyz^2 - 4y^2, 3xy^2 - y^2 z)$, at $(1, -2, 3)$;

   (c) $f(x, y) = (e^x \cos(xy), e^x \sin(xy))$, at $(1, \pi/2)$.

5. Let $u(x, y, z) = f(x^2 + y^2)$. Prove that

$$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0.$$

6. Let $f(x,y,z) = F(u,v,w)$, where $x^2 = vw$, $y^2 = wu$, $z^2 = uv$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w}.$$

Quick Review: Continuity and Differentiability

4 Inverse Function Theorem

Theorem(Inverse function theorem): Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in U$, and $\mathbf{f} : U \to \mathbb{R}^n$ be a function that is continuously differentiable on U. Suppose

$$\det(D\mathbf{f}(\mathbf{a})) \neq 0$$

Then there exists an open set W such that $\mathbf{a} \in W \subseteq U$ and

- $V = \mathbf{f}(W)$ is open;
- the restriction $\mathbf{f}|_W : W \to V$ is bijective i.e. $\mathbf{f}^{-1} : V \to W$ exists;
- $\mathbf{f}^{-1}$ is continuously differentiable on V and $D\mathbf{f}^{-1}(\mathbf{y}) = (D\mathbf{f}(\mathbf{f}^{-1}(\mathbf{y})))^{-1}$ for any $\mathbf{y} \in V$.

Theorem(Mean value inequality): Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable on U such that $\|D\mathbf{f}(\mathbf{a})\| \leq M$ for all $\mathbf{a} \in U$. Suppose $\mathbf{x}, \mathbf{y} \in U$ such that $(1-t)\mathbf{x} + t\mathbf{y} \in U$ for all $t \in [0,1]$, then

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\|$$

Theorem(Contraction mapping theorem): Let $K \subseteq \mathbb{R}^n$ be a closed set and $\mathbf{f} : K \to K$ be a contraction i.e. there exists $C \in (0,1)$ such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq C\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in K$. Then there exists a unique $\mathbf{l} \in K$ such that $\mathbf{f}(\mathbf{l}) = \mathbf{l}$. $\mathbf{l}$ is called a **fixed point** of $\mathbf{f}$.

Problem Set 5

1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$. If $\mathbf{f}(\mathbf{x}, 0)$ is continuous at $\mathbf{x} = 0$ and $\partial_2 \mathbf{f}(\mathbf{x}, \mathbf{y})$ is bounded on $B(0, 1)$. Prove that $\mathbf{f}$ is continuous at $(0, 0)$.

2. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{\pi}{2} + x - \arctan x$$

has no fixed point, and

$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in \mathbb{R}.$$

Note: Why doesn't this example contradict to the contraction mapping theorem?

3. Let differentiable $f : \mathbb{R} \to \mathbb{R}^3$ satisfy $\|f(t)\| = 1, \forall t \in \mathbb{R}$. Show that $\langle Df(t), f(t) \rangle = 0$.

   Note: Can you explain this result geometrically?

4. Find a differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfying that $\exists x, y \in \mathbb{R}^n$ such that

$$f(y) - f(x) \neq Df(a) \cdot (y - x), \forall a \in \mathbb{R}^n.$$

Hint: Use the former problem.

Note: This problem serves as a counterexample to generalize Mean Value Theorem.