MATH 3033 T2A Tutorial 6

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Problem Set 5

2. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{\pi}{2} + x - \arctan x$$

has no fixed point, and

$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in \mathbb{R}.$$

Note: Why doesn't this example contradict to the contraction mapping theorem?

3. Let differentiable $f : \mathbb{R} \to \mathbb{R}^3$ satisfy $\|f(t)\| = 1, \forall t \in \mathbb{R}$. Show that $\langle Df(t), f(t) \rangle = 0$.

   Note: Can you explain this result geometrically?

4. Find a differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfying that $\exists x, y \in \mathbb{R}^n$ such that

$$f(y) - f(x) \neq Df(a) \cdot (y - x), \forall a \in \mathbb{R}^n.$$

Hint: Use the former problem.

Note: This problem serves as a counterexample to generalize Mean Value Theorem.

Quick Review: Continuity and Differentiability

5 Implicit Function Theorem

Theorem 5.1. (Implicit function theorem) Suppose $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m, D \subseteq \mathbb{R}^{n+m}$, and f is continuously differentiable on D and $(a, b) \in D$ such that $f(a, b) = 0$. Let $Df(x, y) = (D_x f(x, y) \quad D_y f(x, y))$, where

$$D_x f = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 & \ldots & \partial_n f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \ldots & \partial_n f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_m & \partial_2 f_m & \ldots & \partial_n f_m \end{pmatrix} \quad \text{and } D_y f = \begin{pmatrix} \partial_{n+1} f_1 & \partial_{n+2} f_1 & \ldots & \partial_{n+m} f_1 \\ \partial_{n+1} f_2 & \partial_{n+2} f_2 & \ldots & \partial_{n+m} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{n+1} f_m & \partial_{n+2} f_m & \ldots & \partial_{n+m} f_m \end{pmatrix}$$

Suppose $\det (D_y f(a, b)) \neq 0$. Then there exists an open set $U \subseteq \mathbb{R}^n$ containing $a, V \subseteq \mathbb{R}^m$ containing b, $U \times V \subseteq D$, and a unique function $g : U \to V$ such that

1. $g(a) = b$,
2. g is continuously differentiable on U,
3. $f(x, g(x)) = 0$ for all $x \in U$, and
4. $Dg(x) = -\left[D_y f(x, g(x))\right]^{-1} D_x f(x, g(x))$ for $x \in U$.

Problem Set 6

1. Polar transformation is
$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$
For $(x, y) \neq (0, 0)$, please calculate $\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{pmatrix}$.

   Hint: Use inverse function theorem.

2. Let

$$\begin{cases} x^2 = vy \\ y^2 = ux. \end{cases}$$

where $(x, y) \neq (0, 0)$. Please calculate $\det\left(\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}\right)$.

Hint: Use inverse function theorem.

3. Let

$$x_1 y_2 - 4x_2 + 2e^{y_1} + 3 = 0,$$
$$2x_1 - x_3 - 6y_1 + y_2 \cos y_1 = 0.$$

Let $f = (f_1, f_2)$ and $y_1 = f_1(x_1, x_2, x_3)$ and $y_2 = f_2(x_1, x_2, x_3)$.

Calculate when $x_1 = -1, x_2 = 1, x_3 = -1, y_1 = 0, y_2 = 1$, the Jocobian matrix $Df(x)$.

Hint: Use implicit function theorem.

4. Let $z = f(x, y), g(x, y) = 0$ and $\dfrac{\partial g}{\partial y} \neq 0$, where f and g are twice differentiable function, please calculate $\dfrac{dz}{dx}$.

   Hint: you can use chain rule or implicit function theorem.