MATH 3033 T2A Tutorial 7

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Quick Review: Sequences and Series of Functions

1 Convergence of Functions

Definition(Pointwise Convergence of Functions): Let $U \subseteq \mathbb{R}^m$ and $f_n : U \to \mathbb{R}$ for $n \in \mathbb{N}$. We say the sequence of functions $\{f_n\}$ converges **pointwise on** U to the limit function $f : U \to \mathbb{R}$ if for each $x \in U, \{f_n(x)\}$ converges and

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Equivalently, for any $\varepsilon > 0$ and $x \in U$, there is an integer N, which may depend on x as well as ε, such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{if } n > N$$

Definition(Uniform Convergence of Functions): Let $U \subseteq \mathbb{R}^m$ and $f_n : U \to \mathbb{R}$ for $n \in \mathbb{N}$. We say the sequence of functions $\{f_n\}$ converges **uniformly on** U to the limit function $f : U \to \mathbb{R}$ if for any $\varepsilon > 0$, there is an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in U \text{ if } n > N$$

In fact, the above condition can also be stated as follows:

$$\|f_n - f\|_U = \sup\{|f_n(x) - f(x)| \mid x \in U\} < \varepsilon \quad \text{if } n > N$$

(In general, for any $g : U \to \mathbb{R}, \|g\|_U = \sup\{|g(x)| \mid x \in U\}$ is called the **sup-norm** of g on U.)

Remark:

1. In definition of pointwise convergence, the integer N may depend on x as well as ε.
2. In definition of uniform convergence, we need that the integer N does not depend on x.

Theorem(L-Test) Let $\{f_n\}$ be a sequence of functions on $U \subseteq \mathbb{R}^m$ such that it converges pointwise to f on U. If there exists a sequence of real numbers $\{L_n\}$ such that for any $n \in \mathbb{N}$,

$$|f_n(x) - f(x)| \leq L_n \quad \text{for all } x \in U$$

and $\lim_{n\to\infty} L_n = 0$, then $\{f_n\}$ converges uniformly to f on U.

Theorem(Uniform Convergence and Continuity): If $\{f_n\}$ converges uniformly to f on U and f_n is continuous on U for all $n \in \mathbb{N}$, f is continuous on U.

Theorem(Cauchy criterion): The sequence of functions $\{f_n\}$, defined on U, converges uniformly on U if and only if for every $\varepsilon > 0$, there exists an integer N such that $n, m \geq N, x \in U$ implies

$$|f_n(x) - f_m(x)| < \varepsilon$$

Definition(Infinite Series of Functions): If $\{f_n\}$ is a sequence of functions defined on $U \subseteq \mathbb{R}^n$, then $\sum_{n=0}^{\infty} f_n$ is said to be an **infinite series of functions on** U. The partial sums of $\sum_{n=0}^{\infty} f_n$ are defined by

$$F_n = \sum_{k=0}^{n} f_k$$

If $\{F_n\}$ converges pointwise to a function F on U, we say that $\sum_{n=0}^{\infty} f_n$ **converges pointwise** to the sum F on U, and write

$$F(x) = \sum_{n=0}^{\infty} f_n(x)$$

If $\{F_n\}$ converges uniformly to a function F on U, we say that $\sum_{n=0}^{\infty} f_n$ **converges uniformly** to the sum F on U.

Theorem(Cauchy criterion for uniform convergence of series): A series $\sum_{n=0}^{\infty} f_n$ converges uniformly on U if and only if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $x \in U$ and $m \geq n > N$,

$$|f_n(x) + f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$$

Theorem(Weierstrass M-test): Suppose $|f_n(x)| \leq M_n$ for all $x \in U$ i.e. f_n is bounded by M_n. Then, if $\sum_{n=0}^{\infty} M_n$ converges, $\sum_{n=0}^{\infty} f_n$ converges uniformly on U.

Definition(Absolute Convergence of Series): $\sum_{n=0}^{\infty} f_n$ is said to converge **absolutely** pointwise/uniformly on U if $\sum_{n=0}^{\infty} |f_n|$ converges pointwise/uniformly on U.

Problem Set 7

1. Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by
$$f_n(x) = xe^{-nx}.$$
Show that (f_n) converges uniformly on $[0, \infty)$.

2. Let $U \subset \mathbb{R}^m$ and $f_n, g_n : U \to \mathbb{R}$ for $n \in \mathbb{N}$. Assume $\{f_n\}$ converges uniformly to f and $\{g_n\}$ converges uniformly to g on U.

   (1) Prove that if both of f and g are bounded on U, then $\{f_n g_n\}$ converges uniformly on U to fg.

   (2) Find an example $\{f_n g_n\}$ does not converge uniformly on U to fg.

3. (Dini's Theorem) Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ such that it converges pointwise to a continuous function f on $[a, b]$. And for any $x_0 \in [a, b]$, $\{f_n(x_0)\}$ is a monotonic sequence. Prove that f_n converges uniformly on $[a, b]$ to f.

4. Let f is continuous on $[0,1]$ and $f(1)=0$. Define $g_n(x)=f(x)x^n$. Show that $\{g_n\}$ uniformly converges on $[0,1]$.