MATH 3033 T2A Tutorial 9

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Quick Review: Sequences and Series of Functions

2 Power Series

Definition(Power Series): Let $a \in \mathbb{R}$ and $\{a_n\}$ be a sequence of real numbers. Then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is called a power series. a is called the center of the power series.

Definition(Limit Superior and Limit Inferior): Given $\{a_n\}$ a sequence of real numbers. The limit superior of $\{a_n\}$ is the limit of $M_n = \sup\{a_m | m \geq n\}$ as $n \to \infty$:

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup\{a_m | m \geq n\} = \lim_{n\to\infty} M_n$$

Similarly, the limit inferior of $\{a_n\}$ is the limit of $m_n = \inf\{a_m | m \geq n\}$ as $n \to \infty$:

$$\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \inf\{a_m | m \geq n\} = \lim_{n\to\infty} m_n$$

It is always true that $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$.

Theorem: Let $H = \lim_{n\to\infty} \sup a_n$ and $h = \lim_{n\to\infty} \inf a_n$ for a bounded sequence $\{a_n\}$. For any $\varepsilon > 0$,

(a) there are infinitely many $n \in \mathbb{N}$ such that $a_n \in (H - \varepsilon, H + \varepsilon)$, and there are at most finitely many $n \in \mathbb{N}$ such that $a_n > H + \varepsilon$;

(b) there are infinitely many $n \in \mathbb{N}$ such that $a_n \in (h - \varepsilon, h + \varepsilon)$, and there are at most finitely many $n \in \mathbb{N}$ such that $a_n < h - \varepsilon$.

Corollary: Let $H = \lim_{n\to\infty} \sup a_n$ and $h = \lim_{n\to\infty} \inf a_n$ for a bounded sequence $\{a_n\}$.

(a) There is a subsequence converging to h and a subsequence converging to H.

(b) If $\mathcal{L} = \{l \in \mathbb{R} | l \text{ is a limit of some convergent subsequence of } \{a_n\}\}$, then h and H are respectively the smallest and the largest element of $\mathcal{L}$.

Theorem(Strong Form of Root Test): Suppose $\{a_n\}$ is a sequence in $\mathbb{R}$.

(a) $\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) $\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Remarks 2.3. Sometimes it is more convenient to use ratio test. In fact, if $a_n \neq 0$ for all $n \in \mathbb{N}$, we have the following inequalities:

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

The above inequalities imply the following **strong form of ratio test**: If $\{a_n\}$ is a sequence such that $a_n \neq 0$ for all $n \in \mathbb{N}$. Then

1. $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely.

2. $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Moreover, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it is equal to $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. (Proof omitted)

Theorem: For any power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, there exists ρ such that $0 \leq \rho \leq +\infty$ and

(a) if $0 < r < \rho$, the series converges absolutely uniformly on $[a-r, a+r]$,

(b) if $|x-a| > \rho$, the series diverges at x.

Remark: ρ is defined by

- **Case 1:** $\frac{1}{\rho} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ if $\{|a_n|^{\frac{1}{n}}\}$ is bounded and $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \neq 0$,
- **Case 2:** $\rho = \infty$ if $\{|a_n|^{\frac{1}{n}}\}$ is bounded and $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$, and
- **Case 3:** $\rho = 0$ if $\{|a_n|^{\frac{1}{n}}\}$ is not bounded above.

Definition: The value of ρ (necessarily unique) in Theorem 2.3 is called the **radius of convergence** of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$.

Remarks:

1. No general statement can be made concerning convergence of a power series at $x = a \pm \rho$ i.e. some series may converge at $x = a \pm \rho$ while some other series may diverge.

2. In general, a power series does not converge uniformly on $(a-\rho, a+\rho)$. For example, $\rho = 1$ for the series $\sum_{n=0}^{\infty} x^n$ and the series converges to $\frac{1}{1-x}$ for $|x| < 1$, diverges for $|x| \geq 1$, and the series does not converge uniformly on $(-1, 1)$.

Theorem: If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $\rho > 0$, then f is differentiable for $|x-a| < \rho$ with

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}.$$

The power series on the right also has radius of convergence ρ.

Corollary: If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $\rho > 0$, then f is infinitely differentiable for $|x-a| < \rho$ with

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x-a)^{n-k}.$$

for any $k \in \mathbb{N}$. In particular, $f^{(k)}(a) = k!a_k$.

Definition: Let $a \in \mathbb{R}$, $\rho > 0$ and $f : (a-\rho, a+\rho) \to \mathbb{R}$ be a function that is infinitely differentiable. The Taylor series of f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The partial sum of the Taylor series up to the k^{th} term is called the **Taylor polynomial** of f of degree k centered at a:

$$\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Remark: If an infinitely differentiable function equals its Taylor series, it is called an **analytic function**.

Theorem 2.5. Let $f : (a-\rho, a+\rho) \to \mathbb{R}$ be a function such that its $(k+1)^{th}$ derivative is continuous at a, where k is a non-negative integer. Then for any $x \in (a-\rho, a+\rho)$,

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{k+1}(x)$$

where

$$\begin{aligned} &\text{(Mean-value form)} \qquad && R_{k+1}(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}, \quad \text{for some } c \text{ between } a \text{ and } x \\ &\text{(Integral form)} \qquad && R_{k+1}(x) = \frac{1}{k!}\int_a^x f^{(k+1)}(t)(x-t)^k dt \end{aligned}$$

Corollary 2.3. Let $f : (a-\rho, a+\rho) \to \mathbb{R}$ be an infinitely differentiable function. Moreover, if there exists $r \in (0, \rho)$ such that

- for any $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f^{(n)}(x)| \leq M_n$ for all $x \in [a-r, a+r]$, and
- $\frac{M_n r^n}{n!} \to 0$ as $n \to \infty$,

then the Taylor series of f centered at a has radius of convergence at least r and converges uniformly to f on $[a-r, a+r]$.

Example 2.2. Special functions like $e^x, \sin x, \cos x$ are analytic functions i.e. they are equal to their own Taylor series because they satisfy the conditions in Corollary 2.3.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

All the above Taylor series have radius of convergence ∞.

Problem Set 9

1. Find the convergence region of each below power series.

   (1) $\sum\limits_{n=1}^{\infty}(nx)^n$;

   (2) $\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}$;

   (3) $\sum\limits_{n=1}^{\infty}x^n$, $\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$, $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^2}$.

2. Consider the following power series centered at 0:

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{5}x + 2x^2 + \frac{1}{5^2}x^3 + 3x^4 + \frac{1}{5^3}x^5 + 4x^6 + \dots$$

(1) Write down the formula for the general term a_n.

(2) Compute $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ and hence the radius of convergence of the power series.

3. Write the Taylor series of functions below centered at $x = 0$ and then specify the interval of convergence.

   (1) $f(x) = \cos^2 x$.

   (2) $f(x) = \ln(1 - x - 2x^2)$.

4. Calculate $\sum_{n=1}^{\infty} \dfrac{n(n+1)}{(2n)!!}$.