MATH 3033 Tutorial Problem Set 11

1. Prove that if $A \in \mathfrak{M}$, then for any $B \subseteq \mathbb{R}$,
$$\mu^*(A \cap B) + \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$
2. Let $E \subseteq \mathbb{R}$. Prove that if $\forall \epsilon > 0$, $\exists A(\epsilon), B(\epsilon) \in \mathfrak{M}$ satisfying $A(\epsilon) \subseteq E \subseteq B(\epsilon)$ and $\mu^*(B(\epsilon) \setminus A(\epsilon)) < \epsilon$, then $E \in \mathfrak{M}$.
3. Find such open set $A \subseteq \mathbb{R}$ that $\mu(A) \neq \mu(\overline{A})$.
4. Let $f(x)$ be a monotonic function. Prove that $f(x)$ is measurable.
5. Let $f_k(x) : [a, b] \to \mathbb{R}$ be measurable functions for $k = 1, 2, \cdots$. Prove that $\exists \{c_k\} \subseteq \mathbb{R}^+$ such that
$$\lim_{k\to\infty} c_k \cdot f_k(x) \text{ exists and is finite a.e. } x \in [a, b].$$
6. Find an example to show that the inequality in the Fatou's lemma can be strict.
7. Denote $[0, 1] \cap \mathbb{Q}$ by $\{r_k\}_{k=1}^{\infty}$. Let $f(x) = \sum_{k=1}^{\infty} \dfrac{1}{k^2 \cdot |x - r_k|^{1/2}}$. Prove that $f(x) < \infty$, a.e. $x \in [0, 1]$.

   Note: $f(x)$ is discontinuous everywhere and unbounded on every interval in $[0, 1]$.
8. Let f be Lebesgue integrable over $[0, \infty)$. Prove that $\lim_{n\to\infty} f(x + n) = 0$ a.e. $x \in [0, \infty)$.