

MATH 3033 Tutorial Problem Set 3

1. Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^2 y^2}{|x|^3 + y^6}.$$

Show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

2. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x \sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

(1) Prove that f is continuous at $(0, 0)$.

(2) Prove that f is not continuous at $(x_0, 0)$ for any $x_0 \neq 0$.

3. Let $F \subset \mathbb{R}$ be closed. Prove that there exists $E \subset F$, such that E is countable and $\bar{E} = F$.

Hint: 1. $\mathbb{Q}$ is countable. 2. If A and B are both countable, then Cartesian product $\{(x, y) | x \in A, y \in B\}$ is countable.

4. Let $F \subset \mathbb{R}$ be closed. Try to design $f : \mathbb{R} \rightarrow \mathbb{R}$, s.t. $F = \{a \in \mathbb{R} | f \text{ is not continuous at } a\}$.

Hint: Use the former question.