MATH 3033 Tutorial Problem Set 5

1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$. If $\mathbf{f}(x, 0)$ is continuous at $x = 0$ and $\partial_2 \mathbf{f}(x, y)$ is bounded on $B((0, 0), 1)$. Prove that $\mathbf{f}$ is continuous at $(0, 0)$.

2. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \frac{\pi}{2} + x - \arctan x$$
has no fixed point, and
$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in \mathbb{R}.$$
Note: Why doesn't this example contradict to the contraction mapping theorem?

3. Let differentiable $f : \mathbb{R} \to \mathbb{R}^3$ satisfy $\|f(t)\| = 1, \forall t \in \mathbb{R}$. Show that $\langle Df(t), f(t) \rangle = 0$.
Note: Can you explain this result geometrically?

4. Find a differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfying that $\exists x, y \in \mathbb{R}^n$ such that
$$f(y) - f(x) \neq Df(a) \cdot (y - x), \forall a \in \mathbb{R}^n.$$
Hint: Use the former problem.
Note: This problem serves as a counterexample to generalize Mean Value Theorem.