MATH 3033 Tutorial Problem Set 7

1. Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by
$$f_n(x) = xe^{-nx}.$$
Show that (f_n) converges uniformly on $[0, \infty)$.

2. Let $U \subset \mathbb{R}^m$ and $f_n, g_n : U \to \mathbb{R}$ for $n \in \mathbb{N}$. Assume $\{f_n\}$ converges uniformly on U to f and $\{g_n\}$ converges uniformly on U to g.
   (1) Prove that if both of f and g are bounded on U, then $\{f_n g_n\}$ converges uniformly on U to fg.
   (2) Find an example $\{f_n g_n\}$ does not converge uniformly on U to fg.

3. (Dini's Theorem) Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ such that it converges pointwise to a continuous function f on $[a, b]$. And for any $x_0 \in [a, b]$, $\{f_n(x_0)\}$ is a monotonic sequence. Prove that f_n converges uniformly on $[a, b]$ to f.

4. Let f is continuous on $[0, 1]$ and $f(1) = 0$. Define $g_n(x) = f(x)x^n$. Show that $\{g_n\}$ uniformly converges on $[0, 1]$.